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FOLIATIONS OF POLYNOMIAL GROWTH ARE HYPERFINITE

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ABSTRACT

We define a class of equivalence relations with polynomial growth and show that such relations always support finite invariant measures and are hyperfinite. In particular, foliations of polynomial growth define hyperfinite equivalence relations with respect to any family of finite invariant measures on transversals. We also extend a result of Dye for countable groups to show that if a locally compact second countable group G acts freely on a Lebesgue space X with finite invariant measure, so that the orbit relation on X is hyperfinite, then G is amenable.

Introduction

The object of this paper is to prove two results about hyperfinite (approximately finite) equivalence relations. In answer to a question of Bowen [1], we show that foliations of polynomial growth define hyperfinite equivalence relations with respect to any finite invariant measure on transversals (§3). In fact we define a class of equivalence relations with polynomial growth and show that such relations always support finite invariant measures and are hyperfinite (§2). In the second part of the paper (§4) we extend a result of Dye [2] about countable groups to show that any locally compact second countable (lcsc) group which has a free action on a Lebesgue space, preserving a finite measure, and such that the orbit equivalence relation is hyperfinite, is necessarily amenable.

NOTATION. All the measures we consider will be σ-finite and Borel. If X is a Lebesgue space, $\mathcal{B}(X)$ denotes the Borel sets of X. Equivalent measures are written $\mu \sim \nu$.

$T_*\mu$ is the measure induced on Y by the mapping $T: X \to Y$; $T_*\mu(A) = \mu(T^{-1}A)$.

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§1. Preliminaries

Let X be a Lebesgue space. An equivalence relation R on X is *Borel* if $R \in \mathscr{B}(X \times X)$ and *countable* (or *finite*) if there are at most countably many (resp. finitely many) points in each equivalence class. We write xRy or $x \sim y$ if $(x, y) \in R$; $\mathrm{orb}_R x$ is the equivalence class of x; and if $E \in \mathscr{B}(X)$, the saturation of E is $[E]_R = \{x \in X : \exists y \in E,\ (x, y) \in R\}$. If $E, F \in \mathscr{B}(X)$, write $R(E, F) = \{T : E \to F$: T is a Borel isomorphism and $(x, Tx) \in R\ \forall x \in E\}$. If R is countable and μ is a measure on X, μ is R *quasi-invariant* if $T_*\mu \sim \mu$ $\forall T \in R(X, X)$. Some care is needed in extending this definition to more general relations; for example, if $X = [0, 1]$ and xRy $\forall x, y \in X$, then if μ is any Borel measure on X, $\exists T \in R(X, X)$ so that $\mu, T_*\mu$ are mutually singular.

A (Borel) *transversal* to R is a set $E \in \mathscr{B}(X)$ such that $\mathrm{orb}_R x \cap E$ is countable, $\forall x \in X$. Let $\mathscr{T}$ be the family of transversals to R. $E \in \mathscr{T}$ is *sufficient* if $[E]_R = X$ (or is conull, if we are concerned with some measure μ on X). R is *concrete* if R has a sufficient transversal. A family of measures $\{\nu_E\}_{E \in \mathscr{T}}$ is *R-quasi-invariant* if

(i) ν_E is a measure on E and $\nu_E(E) > 0$ for some $E \in \mathscr{T}$,

(ii) $T_*\nu_E \sim \nu_F$ whenever $T \in R(E, F)$, $E, F \in \mathscr{T}$.

$\{\nu_E\}_{E \in \mathscr{T}}$ is invariant if $T_*\nu_E = \nu_F$ $\forall T \in R(E, F)$.

PROPOSITION 1.1. *Let R be a concrete equivalence relation with a sufficient transversal E, and let λ be a non-vanishing measure on E such that $T_*\lambda \sim \lambda$ $\forall T \in R(E, E)$. Then there is an R-quasi-invariant family of measures $\{\nu_F\}_{F \in \mathscr{T}}$ with $\nu_E = \lambda$; moreover $\{\nu_F\}_{F \in \mathscr{T}}$ is unique up to measure class, i.e. if $\{\nu'_F\}_{F \in \mathscr{T}}$ has the same properties then $\nu_F \sim \nu'_F$ $\forall F \in \mathscr{T}$.*

PROOF. Suppose $F \in \mathscr{T}$. By repeated use of the von Neumann selection lemma, [13] corollary 8.2, we can find maps $T_i \in R(E, T_iE)$ so that $T_iE \subset F$ and the sets T_iE are disjoint with union F. Define $\nu_{F|T_iE} = T_{i*}\lambda$. The quasi-invariance and essential uniqueness follow since $T_*\lambda \sim \lambda$ $\forall T \in R(E, E)$. Note that if λ is invariant, then so is $\{\nu_E\}_{E \in \mathscr{T}}$.

It is often useful to construct quasi-invariant measures for R starting from a measure μ on X. Two examples of this are given in [11], examples 1.6, 1.7. We recall these briefly:

EXAMPLE 1.2. Let $\mathscr{F}$ be a C^r $(r \geqq 0)$ codimension k foliation of a C^∞ compact n-dimensional manifold M. The leaves of $\mathscr{F}$ define a relation $R_{\mathscr{F}}$ on M

in an obvious way. Let U, ϕ be a co-ordinate chart such that $U \overset{\phi}{\cong} D^k \times D^{n-k}$, where D^r is the open r-disc and $\phi^{-1}(\{x\} \times D^{n-k})$ is the leaf of $\mathcal{F}$ through $\phi^{-1}(x, 0)$, $x \in D^k$. Such a set U is called a distinguished open set, and $K = \phi^{-1}(D^k \times \{0\})$ is a transversal to $\mathcal{F}$. Define ν_k to be the projection of $\mu_{|U}$ along leaves of $\mathcal{F}$. If the family of measures thus obtained is $R_{\mathcal{F}}$ quasi-invariant, $\mathcal{F}$ is said to be ***absolutely continuous*** with respect to μ.

EXAMPLE 1.3. Let G be a lcsc group acting freely and measurable on a Lebesgue space X, with measure μ such that $g_*\mu \sim \mu \, \forall g \in G$. If $U \subset G$ is a neighborhood of the identity and $K \in \mathcal{B}(X)$, UK is a U *flow box* for K if $U \times K \to UK$, $(u, k) \mapsto uk$, is injective, and $\mu(UK) > 0$. Clearly K is a transversal for the orbit relation $R_G = \{(x, gx) \in X \times X; x \in X, g \in G\}$. We will call such transversals *regular*. By [4] theorem 2.8, there are always sufficient transversals for R_G of the form $\bigcup_{i=1}^{\infty} K_i$, where each K_i is a regular transversal. If UK is a flow box and $\pi : UK \to K$ the natural projection, then by Proposition 1.1 above and [11] theorem 3.11, $\{\pi_*\mu_{|UK} : K$ is a regular transversal$\}$ determines an R quasi-invariant family of measures $\{\nu_E\}_{E \in \mathcal{T}}$.

In the case of an invariant measure, we will need the following result:

LEMMA 1.4. *Let G be a* lcsc *group acting freely on a Lebesgue space X, with invariant measure μ. Let $E = \bigcup_{i=1}^{\infty} K_i$ be a sufficient transversal for R_G, where each K_i is a regular transversal with flow box U_iK_i, U_i a neighbourhood of the identity in G, and suppose the sets U_iK_i are disjoint. Let $\pi : X \to E$ be a measurable map with $\pi(x) \in$* orb$_{R_G}x, \mu$ a.a. $x \in X$, *and such that $\pi_{|U_iK_i}$ is the projection to K_i. Let λ be a fixed left Haar measure on G, set $\nu_{K_i} = \lambda(U_i)^{-1}\pi_*\mu_{|U_iK_i}$ and $\nu_E = \sum_{i=1}^{\infty} \nu_{K_i}$. For $b \in E$ and $A \in \mathcal{B}(X)$, let $A(b) = \{g \in G : gb \in A \cap \pi^{-1}(b)\}$. Then*

$$\mu(A) = \int_E \lambda(A(b)) d\nu_E(b).$$

PROOF. Define a measure $\bar{\mu}$ on X by

$$\bar{\mu}(A) = \int_E \lambda(A(b)) d\nu_E(b), \qquad A \in \mathcal{B}(X).$$

By [10] proposition 1.1, $\mu = \bar{\mu}$ on $\bigcup_{i=1}^{\infty} U_iK_i$. Therefore, since $[E]_{R_G} = X$, it will be enough to show that $\bar{\mu}$ is G invariant. By [3] theorem 1, there is a countable group $H = \{h_r\}_{r=1}^{\infty} \subset R(E, E)$ so that $R_H = R_{G|E}$ where $R_{G|E}$ is the restriction of R_G to E. Moreover by [11] theorem 3.11,

$$\frac{dh_* \nu_E(x)}{d\nu_E} = \Delta(g), \quad x \in E, \quad h \in H,$$

where $hx = gx$, $g \in G$, and Δ is the modular function of G.

Fix $g \in G$ and $A \in \mathcal{B}(X)$. Let $A_r = \{a \in A : ga \in \pi^{-1}(h_r \pi(a))\}$. Then $A_r(h_r^{-1}b) = \{g^{-1}kh_r : k \in (gA_r)(b)\}$, and hence

$$\begin{aligned}\bar{\mu}(gA_r) &= \int_E \lambda((gA_r)(b))d\nu_E(b)\\ &= \int_E \lambda(gA_r(h_r^{-1}b)h_r^{-1})d\nu_E(b)\\ &= \int_E \lambda(A_r(h_r^{-1}b))\Delta(h_r)d\nu_E(b)\\ &= \int_E \lambda(A_r(b))d\nu_E(b) = \bar{\mu}(A_r).\end{aligned}$$

Since $A = \bigcup_{r=1}^{\infty} A_r$, this gives the result.

A concrete equivalence relation R on a Lebesgue space X is *hyperfinite* with respect to a family of quasi-invariant measures $\{\nu_E\}_{E \in \mathcal{T}}$ if the countable relation $R_{|E}$ is hyperfinite with respect to ν_E, $\forall E \in \mathcal{T}$; i.e. if there is a sequence of finite relations $S_1 \subset S_2 \subset \cdots \subset X \times X$ on E and with $\mathrm{orb}_{R|_E} x = \bigcup_{n=1}^{\infty} \mathrm{orb}_{S_n} x$ ν_E a.a. $x \in E$ (cf. [4, 11]). Notice that it is in fact sufficient to require that $R_{|E}$ is hyperfinite for one sufficient transversal E, since if F is any other sufficient transversal, there are partitions $E = \bigcup_{i=1}^{\infty} E_i$, $F = \bigcup_{i=1}^{\infty} F_i$, so that $R(E_i, F_i) \neq \phi$.

Suppose R comes from a free group action as in Example 1.3 above. Choose a transversal E and $\pi : X \to E$ as in Lemma 1.4. Let $S_1 \subset S_2 \subset \cdots$ be an increasing sequence of finite relations on E with $\bigcup_{n=1}^{\infty} \mathrm{orb}_{S_n} x = \mathrm{orb}_{R|_E} x$, ν_E a.a. $x \in E$. Define relations R_n, $n = 1, 2, \cdots$, on X by $xR_n y \Leftrightarrow \pi(x) S_n \pi(y)$. Clearly $R_1 \subset R_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} \mathrm{orb}_{R_n} x = \mathrm{orb}_R x$, μ a.a. $x \in X$. We will call the relations R_n obtained in this way *cyclic*; notice that each R_n has a measurable cross-section Z_n which can be taken to be a subset of E of positive ν measure. We call $\nu_{|Z_n}$ the *induced measure* on Z_n. Using Lemma 1.4 and its obvious generalisation to quasi-invariant measures μ (cf. [11] theorem 3.11), it is not hard to show that the relations R_n are cyclic in the sense of [10].

§2. Full groups with polynomial growth

Let R be a countable Borel equivalence relation on a Lebesgue space X. The *full group* of R is $[R]=\{T\in R(A,B):A,B\in\mathcal{B}(X)\}$. $[R]$ is *generated* by $\Gamma\subseteq[R]$ if $\mathrm{orb}_R x=\bigcup_{n=1}^{\infty}\Gamma^n(x)$, where $\Gamma^n(x)=\{y\in X:\ y=\gamma_1\gamma_2\cdots\gamma_i(x),\ \gamma_i\in\Gamma,\ i\leqq n\}$. In what follows we shall always assume Γ is symmetric, i.e. if $\gamma:A\to B\in\Gamma$, then $\gamma^{-1}:B\to A\in\Gamma$. The *growth function* of R at x with respect to Γ is $g_x(n)=|\Gamma^n(x)|$. If

$$\liminf_{n\to\infty}\frac{1}{n}\log g_x(n)>0,$$

R has *exponential growth* with respect to Γ at x. If $g_x(n)$ is dominated by a polynomial of degree d, then R has *polynomial growth* of degree d at x with respect to Γ. It is not hard to see that if R has polynomial growth at x with respect to Γ, then it does so at y, for all $y\in\mathrm{orb}_R x$. Therefore, if R is ergodic with respect to some R quasi-invariant measure μ on X (i.e. the only $[R]$ invariant Borel sets in X are null or conull), then R has polynomial growth either almost everywhere or almost nowhere with respect to a generating set Γ. R has polynomial growth with respect to Γ, if R has polynomial growth at x with respect to Γ $\forall x\in X$ (or μ a.a. $x\in X$). For example, if R is generated by the action of a group of polynomial growth G, then R has polynomial growth with respect to any finite generating set G_0 of G. In particular, every hyperfinite equivalence relation has polynomial growth with respect to a suitable generating set. Notice, however, that a relation does not have polynomial growth with respect to *every* generating set; for example, hyperfinite relations can be generated by the actions of groups with exponential growth, [8].

This definition of polynomial growth is the generalisation to the category of Lebesgue spaces and Borel maps of a pseudo-group with polynomial growth [9]. By analogy with [9] theorem 3.1 we have:

THEOREM 2.1. *Let R be a Borel equivalence relation on a Lebesgue space X, which has polynomial growth with respect to a generating set Γ for R. Then there is an R invariant probability measure μ on X.*

PROOF. It will be sufficient to find μ which is Γ invariant in the sense that $\mu(\gamma A)=\mu(A)$ whenever $\gamma\in\Gamma$ and $A\in\mathcal{B}$ (Domain γ).

We begin by replacing the generating set Γ by a generating set $\Gamma'\subset R(X,X)$.

We can (by thinking of $X\subset[0,1]$) find a countable collection of rectangles

$A_i \times B_j \subset X \times X$ whose union is X minus the diagonal. If $\gamma \in \Gamma$, $\gamma : C \to D$, define

$$\begin{aligned} \gamma_{ij}(x) &= \gamma(x) && x \in A_i \cap C, \\ &= \gamma^{-1}(x) && x \in B_j \cap D, \\ &= x && \text{otherwise,} \end{aligned}$$

Let $\Gamma' = \{\gamma_{ij} : \gamma \in \Gamma\}$. γ_{ij} is a well defined element of $R(X, X)$; moreover since Γ is symmetric $\mathrm{orb}_\Gamma x = \mathrm{orb}_{\Gamma'} x$ $\forall x \in X$. Therefore Γ' generates $[R]$ and R has polynomial growth with respect to Γ'.

Let G be the group generated by Γ'. By [13] theorem 8.7, there is a compact metric G space Y, and an invariant set $E \in \mathcal{B}(Y)$, so that X is G isomorphic to the Borel G subspace of Y defined by E. Γ' generates a pseudo-group of homeomorphisms of Y, of polynomial growth at all points $y \in E$. Since elements of Γ' are defined on all of Y, $f_\gamma(y) = f(\gamma(y))$ is a well defined continuous function whenever $f \in C(Y)$. By the method of [9] theorem 3.1 one constructs a Γ' invariant linear functional I on $C(Y)$ which corresponds to a Γ' invariant probability measure μ on Y with $\mathrm{supp}\, \mu \subset \overline{\mathrm{orb}\, y}$, for each $y \in E$; namely μ is the weak limit of a sequence of measures μ_r, where

$$\int_Y f d\mu_r = \frac{1}{|\Gamma^{n_r}(y)|} \sum_{z \in \Gamma^{n_r}(y)} f(z),$$

for some suitably chosen sequence n_r and $f \in C(Y)$. Choose a decreasing sequence $f_n \in C(Y)$ with $f_n = 1$ on E and $\lim_{n \to \infty} f_n(z) = \chi_E$ μ a.a. $z \in Y$. Then $\mu(E) = \lim_{n \to \infty} \int f_n d\mu$, $\int f_n d\mu = \lim_{r \to \infty} \int f_n d\mu_r$; moreover $\int f_n d\mu_r = 1$ $\forall n, r$ since $\mathrm{orb}_G y \subseteq E$. Therefore $\mu(E) = 1$. μ can therefore be transferred to a Γ invariant measure on X as required.

It was shown in [2] that if R is an equivalence relation generated by the action of a group of polynomial growth, with finite invariant measure, then R is hyperfinite. We generalise this to

THEOREM 2.2. *Let R be a Borel equivalence relation on a Lebesgue space X, which has polynomial growth with respect to a generating set Γ for R. Let μ be an R invariant measure on X. Then R, μ is hyperfinite.*

We divide the proof into a sequence of lemmas.

We begin with the following result, which is essentially the content of [6] corollary 2.2:

PROPOSITION 2.3. *Let $\{an\}_{n=1}^{\infty}$ be a sequence of positive integers dominated by a polynomial in n. Then there are a sequence $\{p_q\}_{q=1}^{\infty}$, $p_q \to \infty$, and a constant $M > 0$, so that*

$$a_{2p_q} < Ma_{p_q}, \forall q \in \mathbf{N} \quad \text{and} \quad \lim_{q\to\infty} a_{p_q-k} a_{p_q}^{-1} = 1, \quad \forall k \in \mathbf{N}.$$

From now on suppose that R is a relation with polynomial growth with respect to Γ, on a Lebesgue space X, and that μ is an R invariant probability measure on X. By the above result, for each $x \in X$, $\exists M(x) > 0$, and $\{P_q(x)\}_{q=1}^{\infty}$, so that

$$|\Gamma^{2P_q(x)}(x)| < M(x)|\Gamma^{P_q(x)}(x)| \quad \text{and} \quad \lim_{q\to\infty} \frac{|\Gamma^{P_q(x)-k}(x)|}{|\Gamma^{P_q(x)}(x)|} = 1, \quad \forall k \in \mathbf{N}.$$

DEFINITION 2.4. Suppose $W \in \mathcal{B}(X)$ and for each $x \in W$, we have $P(x) \subseteq \mathrm{orb}_R x$. If $P(x) \cap P(y) = \emptyset$ whenever $x \neq y$, then $\bar{W} = \bigcup \{P(x) : x \in W\}$ is called the $\{P(x)\}_{x\in W}$ *stack on base* W. A $\{P(x)\}_{x\in W}$ stack $\bar{W}$ has *height* $\leqq N$ if $P(x) \subseteq \Gamma^N(x)$ $\forall x \in W$.

The *n-boundary* of a stack $\bar{W}$ is $\partial_n \bar{W} = \{x \in \bar{W} : \Gamma^n(x) \not\subseteq \bar{W}\}$. A Γ^n stack is a $\{\Gamma^n(x)\}_{x\in W}$ stack on some base W. $P(x)$ is called the *column* of $\bar{W}$ containing x.

LEMMA 2.5. *Suppose $A \in \mathcal{B}(X)$ is such that $|\Gamma^{2n}(x)| < M|\Gamma^n(x)|$ $\forall x \in A$. Then there is a Γ^n stack $\bar{W}$ on base W so that $W \subseteq A$ and $\mu(\bar{W}) > (1/M)\mu(A)$.*

PROOF. By standard techniques it is clear that if $\{\gamma_1(x), \cdots, \gamma_p(x) : \gamma_i \in \Gamma\}$ are distinct for each $x \in B \in \mathcal{B}(X)$, $\mu(B) > 0$, then there is a set $C \in \mathcal{B}(B)$, $\mu(C) > 0$, so that $\gamma_1 C, \cdots, \gamma_p C$ are disjoint. By subdividing A into sets on which precisely $\{\gamma_{i_1}(x), \cdots, \gamma_{i_r}(x) : \gamma_{i_r} \in \Gamma, r \leqq n\}$ are distinct, we see that $C \in \mathcal{B}(A)$, $\mu(C) > 0$, so that C is a base for a Γ^n stack. Choose a maximal such set C. Then $A \subset \Gamma^{2n} C$, otherwise C would not be maximal. Moreover it is clear by a counting argument, using the invariance of μ, that $|\Gamma^{2n} C| < M|\Gamma^n|$. This gives the result.

LEMMA 2.6. *Let $\bar{W} \subseteq X$ be a Γ^n stack on base W and suppose $|\Gamma^n(x)| < (1+\varepsilon)|\Gamma^{n-k}(x)| \forall x \in W$. Then $\mu(\partial_k(\bar{W})) < \mu(\bar{W})$.*

PROOF. $\partial_k \bar{W} \subseteq \bigcup \{\Gamma^n(x) - \Gamma^{n-k}(x) : x \in W\}$. The result follows by a counting argument.

To prove hyperfiniteness we construct stacks of bounded height with small

n-boundary which fill up most of X, for arbitrarily large n. The method is analogous to one of the standard proofs of Rohlin's theorem, cf. [8].

PROPOSITION 2.7. *Suppose* $n \in \mathbf{N}$ *and* $\varepsilon, \delta > 0$ *are given. Then* $\exists Y, Z \in \mathcal{B}(X)$, $N = N(n, \varepsilon, \delta) \in \mathbf{N}$, *and* $\alpha = \alpha(\varepsilon) > 0$ *so that* $Y \subseteq Z$, $\mu(Z) > 1 - \varepsilon$, $\mu(Z - Y) < \delta$, *and so that if* $A \in \mathcal{B}(Y)$ *then there is a stack* $\bar{W}$ *of height* $\leqq N$ *on base* $W \subseteq A$, *with* $\mu(\bar{W}) > \alpha\mu(A)$ *and* $\mu(\partial_n \bar{W}) < \delta\mu(\bar{W})$; *moreover* α *does not depend on* n *or* δ.

PROOF. By Proposition 2.3 we can find $Z \in \mathcal{B}(X)$ and $M > 0$ so that $\mu(Z) > 1 - \varepsilon$ and $M(x) \leqq M$ for $x \in Z$. For $x \in Z$, set

$$p(x) = \min\{p : |\Gamma^{2p}(x)| < M|\Gamma^p(x)| \text{ and } |\Gamma^p(x)| < (1-\delta)|\Gamma^{p-n}(x)|\}.$$

Such p always exist by Proposition 2.3. Let $Y_p = \{x \in Z : p(x) = p\}$. Choose $N \in \mathbf{N}$ so that $\mu(\bigcup_{p=1}^N Y_p) > \mu(Z) - \delta$ and set $Y = \bigcup_{p=1}^N Y_p$. Let $A_p = A \cap Y_p$.

By Lemma 2.5 there is a Γ^N stack $\bar{W}_N$ on base $W_N \subset A_N$, with $\mu(\bar{W}_N) > (1/M)\mu(A_N)$. By Lemma 2.6, $\mu(\partial_n \bar{W}_N) < \delta\mu(\bar{W}_N)$. Repeating the argument, we can inductively find Γ^{N-r} stacks $\bar{W}_{N-r}$ on base $W_{N-r} \subset A_{N-r} - \bigcup_{j > N-r} \Gamma^{2j} W_j$ with

$$\mu(W_{N-r}) > \frac{1}{M}\mu\Big(A_{N-r} - \bigcup_{j>N-r} \Gamma^{2j} W_j\Big) \quad \text{and} \quad \mu(\partial_n \bar{W}_{N-r}) < \delta\mu(\bar{W}_{N-r}).$$

Since $W_i \cap (\bigcup_{j>i} \Gamma^{2j} W_j) = \varnothing$, $\bar{W}_i \cap \Gamma^j W_j = \varnothing\ \forall j > i$. Therefore the sets $\bar{W}_i$ are disjoint and $W = \bigcup_{j=1}^N \bar{W}_j$ is a stack as required, with $\alpha = 1/M$.

PROPOSITION 2.8. *Given* $p \in \mathbf{N}$, $\varepsilon > 0$, *then there are* $N \in \mathbf{N}$ *and a stack* $\bar{V}$ *of height* $\leqq N$ *so that*

$$\mu(\partial_p \bar{V}) < \varepsilon\mu(\bar{V}) \qquad \textit{and} \qquad \mu(\bar{V}) > 1 - \varepsilon.$$

PROOF. Choose α, Z as in Proposition 2.7 using $\varepsilon/2$ in place of ε. Choose $\delta > 0$ so that $(1-\delta)^2(1-\delta-\varepsilon/2) > 1 - \varepsilon$, $\delta(1-\delta)^{-1} < \varepsilon$, and choose m so that $(1-\alpha)^m < \delta$.

Using Proposition 2.7 define N_i, $1 \leqq i \leqq m$, inductively by

$$N_1 = N(2p, \frac{\varepsilon}{2}, \delta), \qquad N_1 \geqq p,$$

and

$$N_i = N(2N_{i-1}, \frac{\varepsilon}{2}, \delta), \qquad N_i \geqq N_{i-1}.$$

Let $Y_i \subseteq Z$, $1 \leqq i \leqq m$, be the sets corresponding to Y in 2.7, so that $\mu(Z - Y_i) < \delta/m$, $1 \leqq i \leqq m$. Let $A = \bigcap_{i=1}^m Y_i$. By 2.7, choose a stack $\bar{W}_m$ of height $\leqq N_m$ on base $W_m \subset A$, with

$$\mu(\bar{W}_m) > \alpha\mu(A) \qquad \text{and} \qquad \mu(\partial_{2N_{m-1}}\bar{W}_m) < \delta\mu(\bar{W}_m).$$

Proceeding inductively choose stacks $\bar{W}_{m-r}$ of height $\leqq N_{m-r}$ on bases $W_{m-r} \subset A - \bigcup_{j>m-r} \bar{W}_j$ with

$$\mu(\bar{W}_{m-r}) > \alpha\mu\Big(A - \bigcup_{j>m-r} \bar{W}_j\Big) \qquad \text{and} \qquad \mu(\partial_{2N_{m-r-1}}\bar{W}_{m-r}) < \delta\mu(\bar{W}_{m-r}).$$

Set $\bar{V}_i = \bar{W}_i - \partial_{N_{i-1}}\bar{W}_i$. Then, since $W_i \subset Y_i - \bigcup_{j>i}\bar{W}_j$, and $N_j \geqq N_i$ for $j > i$,

$$\bar{W}_i \cap \bar{W}_j \subseteq \partial_{N_{j-1}}\bar{W}_j \qquad \text{for } j > i$$

and so $\bar{V}_i \cap \bar{V}_j = \varnothing$, $j > i$.

Set $\bar{V} = \bigcup_{j=1}^m \bar{V}_j$. We have $\mu(\bar{V}) > (1-\delta)\mu(\bigcup_{j=1}^m \bar{W}_j)$ and

$$\mu(\partial_p\bar{V}) < \delta\mu\Big(\bigcup_{j=1}^m \bar{W}_j\Big) < \delta(1-\delta)^{-1}\mu(\bar{V}) < \varepsilon\mu(\bar{V}).$$

Also inductively,

$$\mu\Big(A - \bigcup_{j>i} \bar{W}_m\Big) < (1-\alpha)^{m-j+1}\mu(A), \qquad 1 \leqq j \leqq m.$$

Therefore

$$\mu(\bar{V}) > (1-\delta)\mu\Big(\bigcup_{j=1}^m \bar{W}_j\Big) > (1-\delta)^2\mu(A) > (1-\delta)^2\Big(1-\delta-\frac{\varepsilon}{2}\Big) > 1-\varepsilon.$$

Therefore, $\bar{V}$ is a stack of height $\leqq N_m$ as required.

Proof of Theorem 2.2. Suppose inductively that finite relations $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n$ on X and integers $M(1) \leqq M(2) \leqq \cdots \leqq M(N)$ have been chosen such that

(1) $$\mathrm{orb}_{R_j}(x) \subseteq \mathrm{orb}_R(x), \qquad \forall x \in X,$$

(2) $$\mu\{x \in X : \Gamma^j(x) \not\subseteq \mathrm{orb}_{R_j}(x)\} < \frac{\varepsilon}{2^j}, \qquad j \leqq n,$$

(3) $$xR_jy \Rightarrow x \in \Gamma^{M(j)}(y), \qquad j \leqq n.$$

By Proposition 2.8 find a stack $\bar{W}$ of height $\leqq N$ such that

$$\mu(\partial_{M(n)}\bar{W}) < \frac{\varepsilon}{2^{n+2}}\mu(\bar{W}) \quad \text{and} \quad \mu(\bar{W}) > 1 - \frac{\varepsilon}{2^{n+2}}.$$

Set $\bar{V} = \bar{W} - \partial_{M(n)}\bar{W}$. Define R_{n+1} as follows: $xR_{n+1}y$ if and only if either

(a) $x, y \notin [\bar{V}]_{R_n}$, xR_ny

or

(b) $x, y \in [\bar{V}]_{R_n}$, xR_nw, yR_nz, and w, z are in the same column of $\bar{V}$.

Setting $M(n+1) \geqq \max(2N, M(n))$, it is not hard to check that R_{n+1} satisfies (1), (2) and (3) and that R is therefore hyperfinite.

§3. Foliations of polynomial growth

Foliations of polynomial growth are defined in [9]. Let $\mathcal{F}$ be a C^r foliation of a compact manifold M and let $\{U_i\}_{i=1}^n$ be a cover of M by distinguished open sets with coordinate maps $U_i \overset{\phi_i}{\cong} D^k \times D^{n-k}$ as in Example 1.2. Let $K_i = \phi_i^{-1}(D^k \times \{0\})$ and let $\mathcal{P}_i(x) = \phi_i^{-1}(\{\phi_i(x)\} \times D^{n-k})$, $x \in K_i$. One can choose $\{u_i\}_{i=1}^n$ so that, if $x \in U_i$, $\mathcal{P}_i(x) \cap \mathcal{P}_j(y) \neq \emptyset$ for at most one $y \in K_j$. Let $X = \bigcup_{i=1}^n K_i$. Define partial homeomorphisms $\gamma_{ij} : K_i \to K_j$, $\gamma_{ij}(x) = y$ if and only if $\mathcal{P}_i(x) \cap \mathcal{P}_j(y) \neq \emptyset$. $\Gamma = \{\gamma_{ij} : 1 \leqq i, j \leqq n\}$ generates a pseudogroup of homeomorphisms of X. $\mathcal{F}$ has polynomial growth if $\forall x \in X$, $g_x(n)$ is bounded by a polynomial in n, where

$$g_x(n) = |\{y \in X : y = \gamma_{i_1i_2}\gamma_{i_2i_3}\cdots\gamma_{i_ri_{r+1}}(x), r \leqq n\}|.$$

It is clear that X is a sufficient transversal for $R_{\mathcal{F}}$ in the sense of §1, and that $R_{\mathcal{F}|X}$ has polynomial growth with respect to the set Γ. Therefore by Theorem 2.1, there is an $R_{\mathcal{F}|X}$ invariant measure on X, and hence by Proposition 1.1 a family of $R_{\mathcal{F}}$ invariant measures on transversals to $R_{\mathcal{F}}$. Moreover any $R_{\mathcal{F}}$ is hyperfinite with respect to any $R_{\mathcal{F}}$ invariant family of measures on transversals. (If the measure on X is infinite, we simply partition X into subsets of finite measure and work on each set separately.)

Remark 3.1. It would be desirable to have this result without restriction to the case of invariant measure; unfortunately the methods of §2 do not seem to be adequate. Note, however, the following facts:

(i) The stable and unstable foliations of Anosov diffeomorphisms have polynomial growth by [12] lemma 4.

(ii) These foliations are absolutely continuous with respect to any equilibrium state, and hyperfinite with respect to the induced measure on transversals (which is not in general invariant), [11, 1].

§4. Hyperfiniteness and amenability

It was shown in [2] p. 159 that if G is a countable group acting freely on a Lebesgue space X with finite invariant measure μ so that R_G is hyperfinite, then G is amenable. We extend this result to arbitrary lcsc groups. Our method gives a different proof of Dye's result in the countable case, and is related to ideas in [7].

We shall show G is amenable by finding highly invariant sets $A \subseteq G$. More precisely, we use the following condition for amenability, [5] p. 65:

(*) Given $\varepsilon, \delta > 0$ and a compact set $K \subseteq G$, there exist $A \in \mathcal{B}(G)$, $N \in \mathcal{B}(K)$, with $0 < \lambda(A) < \infty$, $\lambda(N) < \delta$, and $\lambda(xA \Delta A) < \varepsilon\lambda(A)\ \forall x \in K - N$, where λ is a fixed left Haar measure on G.

To build highly invariant sets A, we find a cyclic relation $R_n \subset R_G$ such that for most $x \in X$, $\{g \in G : gx \in R_n x\}$ is a highly invariant set in G. In fact we shall show that G satisfies the following condition:

(**) Given $\varepsilon > 0$ and a compact set $K \subseteq G$, there exists a relatively compact set $F \subset G$ such that

$$\lambda \times \lambda(\{(k, f) \in K \times F : kf \notin F\}) < \varepsilon\lambda(K)\lambda(F).$$

Lemma 4.1. *Condition* (**) *implies* (*).

Proof. Suppose (**) holds. Given $\varepsilon, \delta > 0$, $K \subseteq G$ compact, find F such that

$$\lambda \times \lambda(\{(k, f) \in K \times F\colon kf \notin F\}) < \frac{\delta\varepsilon}{2}\lambda(F).$$

For $k \in K$, set $F_k = \{f \in F : kf \in F\}$, and set

$$C = \{k \in K : \lambda(F - F_k) > \frac{\varepsilon}{2}\lambda(F)\}.$$

Then

$$\lambda(C)\lambda(F)\frac{\varepsilon}{2} \leqq \int_C \lambda(F - F_k)d\lambda(k) < \frac{\varepsilon}{2}\delta\lambda(F)$$

so that $\lambda(C) < \delta$. Clearly $F_k \subset F$, $kF_k \subset F$ and $\lambda(F_k) > (1 - \varepsilon/2)\lambda(F)$. Therefore

$$\lambda(kF - F) \leqq \lambda(kF - kF_k) = \lambda(F - F_k)$$

and

$$\lambda(F - kF) \leqq \lambda(F - kF_k) = \lambda(F - F_k)$$

since

$$\lambda(F - F_k) + \lambda(F_k) = \lambda(F) = \lambda(F - kF_k) + \lambda(kF_k) = \lambda(F - kF_k) + \lambda(F_k).$$

Hence $\lambda(kF\Delta F) < \varepsilon\lambda(F)$ for $k \not\in C$.

LEMMA 4.2. *Let G be a* lcsc *group acting on the Lebesgue space X, with finite invariant measure μ, so that R_G is hyperfinite. Then there are cyclic relations $R_1 \subseteq R_2 \subseteq \cdots$, so that $R_G = \bigcup_{n=1}^{\infty} R_n$ and so that* $\operatorname{orb}_{R_n} x \subseteq Kx$ *for all $x \in X$, where $K \subseteq G$ is a compact set depending on n but not on x.*

PROOF. Choose a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact sets with $\bigcup_{n=1}^{\infty} K_n = G$. Suppose $S_1 \subseteq S_2 \subseteq \cdots$ is an increasing sequence of cyclic relations on X with $\bigcup_{n=1}^{\infty} S_n = R_G$. Let Z_n be a measurable cross-section for S_n. Let $A_n^m = \{x \in X: xS_n z, z \in Z_n, x = hz, h \in K_m\}$. Then $\bigcup_{m=1}^{\infty} A_n^m = X$ for each n, and $\exists m(n)$ such that $\mu(A_n^m) > 1 - 2^{-n}$. We may clearly assume $m(1) \leqq m(2) \leqq \cdots$.

Let $C_n = (A_n^{m(n)})'$ and let $B_n = \bigcup_{k=n}^{\infty} C_k$. Define R_N, $N = 1, 2, \cdots$, by

$$R_{N|X-B_N} = S_N, \qquad R_{N|B_N} = \Phi,$$

where Φ is the trivial relation with one point equivalence classes. It is clear that on $(\bigcap_{n=1}^{\infty} B_n)'$, $R_1 \subseteq R_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} R_n = R_G$. Moreover, $\mu(\bigcap_{n=1}^{\infty} B_n) = 0$, and $\operatorname{orb}_{R_n} x \subseteq K_{m(n)} K_{m(n)}^{-1} x$, $\forall x \in X$.

THEOREM 4.3. *Let G be a* lcsc *group acting freely and measurably on a Lebesgue space X, with finite invariant measure μ, so that R_G is hyperfinite. Then G satisfies* (**) *(and hence is amenable).*

PROOF. Since G acts freely on X, the map $(g, x) \mapsto (gx, x)$ identifies $G \times X$ with $R_G \subset X \times X$. Let ω be the measure $\lambda \times \mu$ induced on R_G, where λ is a fixed left Haar measure on G.

Suppose a compact $K \subset G$ and $\varepsilon > 0$ are given. Let $R_1 \subset R_2 \subset \cdots$ be an increasing sequence of cyclic relations chosen to satisfy the conditions of Lemma 4.2, with $\bigcup_{n=1}^{\infty} R_n = R_G$.

Since $\omega(K \times X) = \lambda(K)\mu(X) < \infty$, $\exists n \in \mathbf{N}$ such that

$$\omega(K \times X \cap R_n) > (1-\varepsilon^3)\omega(K \times X).$$

For $x \in X$, let $C(x) = \{g \in K : gx \not\in \mathrm{orb}_{R_n} x\}$ and let $Y = \{x \in X : \lambda(C(x)) < \varepsilon\lambda(K)\}$. Then

$$\begin{aligned}(1-\varepsilon^2)\omega(K \times X) &< \omega(K \times X \cap R_n)\\ &= \int_Y \lambda(K - C(x))d\mu(x) + \int_{X-Y} \lambda(K - C(x))d\mu(x)\\ &\leqq \lambda(K)\mu(Y) + (1-\varepsilon)\lambda(K)\mu(X - Y),\end{aligned}$$

so that

$$1 - \varepsilon^2 \leqq \mu(Y). \tag{4.4}$$

Now let $Z \in \mathcal{B}(X)$ be a cross-section for R_n, and for $z \in Z$, set $F(z) = \{g \in G : gz \in R_n z\}$. By assumption $F(z)$ is contained in a compact set and so $0 < \lambda(F(z)) < \infty$ for $z \in Z^*$, where $Z^* \in \mathcal{B}(Z)$ and $[Z^*]_{R_n}$ is conull. Let $Y(z) = \{g \in G : gz \in Y\}$ and let

$$Z_1 = \{z \in Z : \lambda(F(z) \cap Y(z)) < (1-\varepsilon)\lambda(F(z))\}.$$

By Lemma 1.4,

$$\begin{aligned}\mu([Z_1]_{R_n} \cap Y) &= \int_{Z_1} \lambda(F(z) \cap Y(z))d\nu(z)\\ &< (1-\varepsilon)\int_{Z_1} \lambda(F(z))d\nu(z)\\ &= (1-\varepsilon)\mu([Z_1]_{R_n})\end{aligned}$$

where ν is the induced measure on Z. By (4.4),

$$\begin{aligned}1-\varepsilon^2 \leqq \mu(Y) \leqq \mu(Y \cap [Z_1]_{R_n}) + \mu(X - [Z_1]_{R_n})\\ < (1-\varepsilon)\mu([Z_1]_{R_n}) + \mu(X - [Z_1]_{R_n}),\end{aligned}$$

hence

$$\mu([Z_1]_{R_n}) < \varepsilon.$$

Choose $z_0 \in Z - Z_1$ with $0 < \lambda(F(z_0)) < \infty$, and set $F = F(z_0)$, $F' = \{g \in F : gz_0 \in Y\}$. Then

$$\{(k,f) \in K \times F : kf \not\in F\} \subseteq \left(\bigcup_{g \in F'} (C(gz_0) \times \{g\})\right) \cup (K \times (F - F')).$$

$$\lambda \times \lambda(K \times (F - F')) \leqq \lambda(K)\lambda(F - F') \leqq \varepsilon\lambda(K)\lambda(F)$$

and

$$\lambda \times \lambda\Big(\bigcup_{g \in F'} C(gz_0) \times \{g\}\Big) = \int_{F'} \lambda(C(gz_0))d\lambda(g) < \varepsilon\lambda(K)\lambda(F').$$

Thus $\lambda \times \lambda(\{(k,f) \in K \times F : kf \not\in F\}) \leqq 2\varepsilon\lambda(K)\lambda(F)$, as required.

Note added in proof. Since writing this paper the work of M. Samuelides, *Tout feuilletage à croissance polynomiale est hyperfini*, Publications Mathématiques de l'Université Pierre et Marie Curie, No. 10, 1978, which contains a similar result to our §3, has been brought to our attention.

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